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Inequalities involving modified Bessel functions of the first kind II

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Abstract

The intrinsic properties, including logarithmic convexity (concavity), of the modified Bessel functions of the first kind and some other related functions are obtained. Several inequalities involving functions under discussion are established.

Keywords: Modified Bessel functions of the first kind; Logarithmic convexity (concavity); Inequalities

1. Introduction

This paper deals with the modified Bessel functions of the first kind and some other related functions. Emphasis is on intrinsic properties such as logarithmic convexity (logarithmic concavity), monotonicity and the inequalities obeyed by the functions under discussion.

The modified Bessel function of the first kind of order p, denoted by $I_p(x)$, is a particular solution of the second-order differential equation [10, p. 77]

$$x^2y''(x) + xy'(x) - \left(x^2 + p^2\right)y(x) = 0 \tag{1.1}$$

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and is represented explicitly by the infinite series as

$$I_p(x)=\sum_{n=0}^{\infty}\frac{1}{n!\Gamma(p+n+1)}\left(\frac{x}{2}\right)^{2n+p} \tag{1.2}$$

($x\in\mathbb{R}$, $p\neq -1,-2,\ldots$). Another function studied in this paper is defined as follows

$$\mathcal{I}_p(x)=2^p\Gamma(p+1)x^{-p}I_p(x) \tag{1.3}$$

($x\in\mathbb{R}$, $p\neq -1,-2,\ldots$). Clearly

$$\mathcal{I}_p(x)=\sum_{n=0}^{\infty}\frac{(1/4)^n}{(p+1)_n n!}x^{2n}, \tag{1.4}$$

where $(a)_n=\Gamma(a+n)/\Gamma(a)$ ($a\neq 0,-1,\ldots$, $n\in\mathbb{N}$) stands for the Pochhammer symbol.

On several occasions we shall use a function γ_p, where

$$\gamma_p(x)=\mathcal{I}_p(\sqrt{x}\,). \tag{1.5}$$

We have

$$\gamma_p(x)=\sum_{n=0}^{\infty}\frac{(1/4)^n}{(p+1)_n n!}x^{n} \tag{1.6}$$

and

$$\gamma_p(x)=2^p\Gamma(p+1)x^{-p/2}I_p(\sqrt{x}\,). \tag{1.7}$$

The following function

$$v_p(x)=x\frac{I_p(x)}{I_{p+1}(x)} \tag{1.8}$$

is of interest in finite elasticity. For more details the interested reader is referred to [9] and the references therein. For later use, let us record a formula

$$v_p(x)=2(p+1)\frac{\gamma_p(x^2)}{\gamma_{p+1}(x^2)} \tag{1.9}$$

which is an immediate consequence of (1.8) and (1.7).

This paper is a continuation of our earlier work [7]. The main results are contained in the next section. They provide either enhancements of some results derived in [7] (see Theorem 2.1) or they contain complementary results (see Theorem 2.2) to those included in Theorem 2.1.

2. Main results

Before we state and prove the main results of this paper, let us recall a lemma which will be used in the sequel.

Lemma 2.1. [8] *Suppose that the power series*

$$f(x)=\sum_{n=0}^{\infty}\alpha_n x^n \quad \textit{and} \quad g(x)=\sum_{n=0}^{\infty}\beta_n x^n$$

($\beta_n>0$ *for all* $n\geqslant 0$) *both converge for* $|x|<\infty$. *Then the function* $f(x)/g(x)$ *is* (*strictly*) *increasing* (*decreasing*) *for* $x>0$ *if the sequence* $\{\alpha_n/\beta_n\}_{n=0}^{\infty}$ *is* (*strictly*) *increasing* (*decreasing*).

It is worth mentioning that this lemma was used, among other things, to prove many interesting inequalities for the zero-balanced Gauss hypergeometric function (see [3,8]) and also for the generalized Bessel functions (see [4–6] for more details).

In what follows, the symbols G, A, and A_2 will stand, respectively, for the unweighted geometric, arithmetic, and the second-order power means of the positive numbers x and y, i.e.,

$$G \equiv G(x,y) = \sqrt{xy}, \qquad A \equiv A(x,y) = \frac{x+y}{2}, \qquad A_2 \equiv A_2(x,y) = \sqrt{\frac{x^2+y^2}{2}}.$$

It is well known that $G < A < A_2$ provided $x \neq y$. Our first result reads as follows.

Theorem 2.1. *Let the numbers a and b be such that $a > 1$ and $b = 1/(4\ln a)$. If $p \geqslant b-1$ and $0 < x \leqslant 2b$, then*

(a) *the function $\mathcal{I}_p(x)$ is strictly log-convex in x;*
(b) *the function $I_p(x)$ is strictly log-convex in x provided $a \geqslant \exp(1/4)$ and $p \leqslant 0$;*
(c) *the following inequalities*

$$I_p(\alpha x) < \alpha^p I_p(x)\big[\mathcal{I}_p(x)\big]^{\alpha-1} \quad (0 < \alpha < 1), \tag{2.1}$$

$$\big[xI_p'(x)\big]^2 < pI_p^2(x) + x^2 I_p(x) I_p''(x), \tag{2.2}$$

$$\frac{I_p(A)}{\sqrt{I_p(x)I_p(y)}} < \left(\frac{A}{G}\right)^p \quad (x \neq y,\ 0 < y \leqslant 2b), \quad \textit{and} \tag{2.3}$$

$$v_{p-1}^2(x) - (2p-1)v_{p-1}(x) - x^2 < 2p \tag{2.4}$$

are valid.

Let us note that for $p > -\frac{1}{2}$ and $x > 0$, Theorem 2.1 has been established in [7]. Since the proofs of the inequalities (2.1)–(2.4) go along the lines introduced in [7], they are not included in this paper.

Proof of parts (a) and (b) of Theorem 2.1. (a) In order to prove that the function $\mathcal{I}_p(x)$ is strictly log-convex it suffices to show that its logarithmic derivative $[\ln \mathcal{I}_p(x)]' = \mathcal{I}_p'(x)/\mathcal{I}_p(x)$ is a strictly increasing function. Using (1.5) we obtain $\mathcal{I}_p'(x)/\mathcal{I}_p(x) = 2x\gamma_p'(x^2)/\gamma_p(x^2)$. It is clear that the last function is strictly increasing if the function $\phi(x) := \sqrt{x}\gamma_p'(x)/\gamma_p(x)$ is strictly increasing. To prove this let us represent function ϕ as the product of three functions:

$$\phi(x) = \frac{\sqrt{x}}{a^x} \cdot \frac{a^x}{\gamma_p(x)} \cdot \gamma_p'(x)$$

($a > 1$, $x > 0$). One can easily verify that the function $\sqrt{x}/a^x$ is an increasing function provided $0 < x \leqslant 1/(2\ln a) = 2b$. It follows from (1.6) that $\gamma_p'(x)$ is strictly increasing on its domain. To complete the proof of monotonicity of the function ϕ it suffices to show that the function $a^x/\gamma_p(x)$ is increasing for $x > 0$. Using series expansion for a^x and utilizing (1.6) we obtain

$$\frac{a^x}{\gamma_p(x)} = \frac{\sum_{n=0}^{\infty} \alpha_n x^n}{\sum_{n=0}^{\infty} \beta_n x^n},$$

where $\alpha_n = (\ln a)^n/n!$ and $\beta_n = (1/4)^n/[(p+1)_n n!]$ for $n \geqslant 0$. Clearly $\beta_n > 0$ for all n. Using Lemma 2.1, we see that the sequence $\{A_n\}_{n=0}^{\infty}$, where $A_n = \alpha_n/\beta_n$ is increasing because

$$A_{n+1}/A_n = (4\ln a)(p+n+1) = (p+n+1)/b \geqslant b/b = 1.$$

(b) Using (1.3), we obtain $I_p(x) = (x/2)^p \mathcal{I}_p(x)/\Gamma(p+1)$. Function $x \to x^p$ is log-convex for $p \leqslant 0$ while the function $x \to \mathcal{I}_p(x)$ is strictly log-convex for $b-1 \leqslant p \leqslant 0$ by part (a). Thus the product in question is a strictly log-convex function in x for all $p \in [b-1, 0]$. Assumption $a \geqslant \exp(1/4)$ implies that the interval $[b-1, 0]$ is not empty. The proof is complete. □

The next result reads as follows.

Theorem 2.2. *Let $p > -1$ and let $x > 0$. Then*

(a) *the function $\gamma_p(x)$ is strictly log-concave in x*;
(b) *the function $I_p(\sqrt{x})$ is strictly log-concave in x provided $p > 0$*;
(c) *the function $I_p(x)$ is strictly log-concave in x provided $p > 0$ and $x \leqslant \sqrt{3}p$*;
(d) *the function $v_{p-1}(x)$ is strictly increasing*;
(e) *the following inequalities*

$$I_p(\alpha x) > \alpha^p I_p(x)\big[\mathcal{I}_p(x)\big]^{\alpha^2-1} \quad (0 < \alpha < 1), \tag{2.5}$$

$$\big[xI_p'(x)\big]^2 > 2pI_p^2(x) - xI_p(x)I_p'(x) + x^2 I_p(x)I_p''(x), \tag{2.6}$$

$$\frac{I_p(A_2)}{\sqrt{I_p(x)I_p(y)}} > \left(\frac{A_2}{G}\right)^p \quad (x \neq y,\ y > 0), \tag{2.7}$$

$$1 + xI_{p-1}'(x)/I_{p-1}(x) > xI_p'(x)/I_p(x), \tag{2.8}$$

$$v_{p-1}^2(x) - 2(p-1)v_{p-1}(x) - x^2 > 4p \tag{2.9}$$

hold true.

Proof. (a) In order to establish the assertion it suffices to prove that the logarithmic derivative of $\gamma_p(x)$ is a strictly decreasing function of x. To this aim we will employ Lemma 2.1. Making use of (1.6) we obtain

$$\frac{\gamma_p'(x)}{\gamma_p(x)} = \sum_{n=0}^{\infty} \frac{(1/4)^{n+1}}{(p+1)_{n+1} n!} x^n \Big/ \sum_{n=0}^{\infty} \frac{(1/4)^n}{(p+1)_n n!} x^n.$$

Utilizing notation used in Lemma 2.1 we obtain

$$A_n := \alpha_n/\beta_n = \frac{1}{4(p+n+1)}, \quad n \geqslant 0.$$

This in turn implies that $A_n > A_{n+1}$ because $p > -1$.

(b) It follows from (1.7) that

$$I_p(\sqrt{x}) = \frac{1}{2^p \Gamma(p+1)} x^{p/2} \gamma_p(x).$$

Thus the function $I_p(\sqrt{x})$ is strictly log-concave in x as the product of the log-concave function $x \to x^{p/2}$ and the strictly log-concave function $\gamma_p(x)$. Hence the assertion follows.

(c) In order to prove that the modified Bessel function of the first kind $I_p(x)$ is strictly log-concave in x when $p>0$ and $0<x\leqslant\sqrt{3}p$, it suffices to show that its logarithmic derivative $I_p'(x)/I_p(x)$ is strictly decreasing, i.e., that $I_p(x)I_p''(x)-[I_p'(x)]^2<0$. Making use of the inequality (2.6) we obtain

$$I_p(x)I_p''(x)-\left[I_p'(x)\right]^2<\frac{1}{x}I_p(x)\left[I_p'(x)-2p\frac{1}{x}I_p(x)\right].$$

In order to complete the proof of part (c) it suffices to show that

$$xI_p'(x)/I_p(x)<2p. \tag{2.10}$$

Using Amos' inequality [2, p. 243]

$$I_{p-1}(x)I_{p+1}(x)-I_p^2(x)<0 \tag{2.11}$$

($x>0$, $p\geqslant 0$) together with two formulas $I_{p-1}(x)=(p/x)I_p(x)+I_p'(x)$ and $I_{p+1}(x)=I_p'(x)-(p/x)I_p(x)$ (see [1]) the inequality (2.11) can be written as

$$xI_p'(x)/I_p(x)<\sqrt{x^2+p^2}.$$

Since $0<x\leqslant\sqrt{3}p$, $xI_p'(x)/I_p(x)<2p$ which completes the proof of (2.10).

(d) Making use of (1.9), we obtain

$$v_p(\sqrt{x})=2(p+1)\frac{\gamma_p(x)}{\gamma_{p+1}(x)}.$$

We use Lemma 2.1 again with

$$\alpha_n=\frac{2(p+1)(1/4)^n}{(p+1)_n n!}\quad\text{and}\quad\beta_n=\frac{(1/4)^n}{(p+2)_n n!},\quad n\geqslant 0,$$

to conclude that the sequence $\{\alpha_n/\beta_n\}_{n=0}^{\infty}$, where

$$\frac{\alpha_n}{\beta_n}=2(p+n+1)$$

is strictly increasing, hence the function $v_{p-1}(\sqrt{x})$ is strictly increasing. This in turn implies that the assertion (d) holds true.

(e) Since $\gamma_p(x)$ is strictly log-concave, the inequality

$$\gamma_p\big(\alpha x+(1-\alpha)y\big)>\left[\gamma_p(x)\right]^{\alpha}\left[\gamma_p(y)\right]^{1-\alpha} \tag{2.12}$$

($x\neq y$) holds true for $0<\alpha<1$ and $p>-1$. Letting $y=0$ and taking into account that $\gamma_p(0)=1$ we obtain $\gamma_p(\alpha x)>[\gamma_p(x)]^{\alpha}$. This in conjunction with (1.7) gives

$$I_p(\sqrt{\alpha x})>(\sqrt{\alpha})^p I_p\left[\gamma_p(x)\right]^{\alpha-1}.$$

Replacing x by x^2, α by α^2, and utilizing (1.5) we obtain the inequality (2.5). For the proof of the inequality (2.6) we appeal again to the logarithmic concavity of the function $\gamma_p(x)$. This in turn implies that the function $\gamma_p'(x)/\gamma_p(x)$ is decreasing. Utilizing (1.7), one obtains

$$\frac{\gamma_p'(x)}{\gamma_p(x)}=\frac{I_p'(\sqrt{x})}{2\sqrt{x}I_p(\sqrt{x})}-\frac{p}{2x}.$$

Hence

$$\left[\frac{\gamma_p'(x)}{\gamma_p(x)}\right]' = \frac{I_p''(\sqrt{x}\,)}{4xI_p(\sqrt{x}\,)} - \frac{I_p'(\sqrt{x}\,)}{4x\sqrt{x}I_p(\sqrt{x}\,)} - \frac{1}{4x}\left[\frac{\gamma_p'(x)}{\gamma_p(x)}\right]^2 + \frac{p}{2x^2} < 0.$$

Replacing x by x^2 completes the proof of (2.6). In order to establish the inequality (2.7) we let $\alpha = 1/2$ in (2.12) and next apply (1.7) to obtain

$$I_p^2\left(\sqrt{\frac{x+y}{2}}\right) > \left(\frac{x+y}{2\sqrt{xy}}\right)^p I_p(\sqrt{x}\,)I_p(\sqrt{y}\,).$$

Replacing x by x^2 and y by y^2 we obtain the assertion (2.7). Inequality (2.8) is an immediate consequence of the property of $v_{p-1}(x)$ established in part (d). This in conjunction with (1.8) gives

$$\left[xI_{p-1}(x)/I_p(x)\right]' > 0.$$

The assertion (2.8) now follows. For the proof of (2.9) we divide both sides of (2.6) by $I_p(x)$ to obtain

$$\left[xI_p'(x)\right]^2/I_p(x) + xI_p'(x) - 2pI_p(x) > x^2 I_p''(x)$$

($x > 0$, $p > -1$). Combining this with the differential equation (1.1) we obtain

$$h_p^2(x) + h_p(x) - 2p > x^2 + p^2 - h_p(x), \tag{2.13}$$

where $h_p(x) = xI_p'(x)/I_p(x)$. Taking into account that $xI_p'(x) = xI_{p-1} - pI_p(x)$ (see, e.g., [1, Eq. (9.6–26)]), we obtain $h_p(x) = v_{p-1}(x) - p$. Substituting this into (2.13) gives the desired result. This completes the proof. □

Corollary 2.1. *Let the numbers a and b be such that $1 < a < \exp(1/4)$ and $b = 1/(4\ln a)$. If $0 < x, y \leqslant 2b$ and $p \geqslant b - 1$, then the following inequalities*

$$\frac{I_p(A)}{\sqrt{I_p(x)I_p(y)}} < \left(\frac{A}{G}\right)^p < \left(\frac{A_2}{G}\right)^p < \frac{I_p(A_2)}{\sqrt{I_p(x)I_p(y)}} \tag{2.14}$$

are valid.

Proof. With $a < \exp(1/4)$ one has $p \geqslant b - 1 > 0$. Making use of (2.3) and (2.7), we obtain the desired result (2.14). □

Before we state the next corollary, let us introduce a two-parameter function $v_{p,q}(x)$, where

$$v_{p,q}(x) = x\frac{I_p(x)}{I_q(x)}. \tag{2.15}$$

Let us note that $v_{p,p+1}(x) = v_p(x)$.

We have the following.

Corollary 2.2. *Let $a > \exp(1/4)$ and $b = 1/(4\ln a)$. If $p \geqslant 2b/\sqrt{3}$ and $b - 1 \leqslant q \leqslant 0$, then the function $v_{p,q}(x)$ is strictly log-concave in x for $0 < x \leqslant 2b$.*

Proof. We shall establish logarithmic concavity of the function $v_{p,q}(x)$ by showing that each of the following functions x, $I_p(x)$, and $1/I_q(x)$ is log-concave in x. Clearly the identity function $x \to x$ is log-concave for all $x > 0$. Using part (c) of Theorem 2.2 we see that the function $I_p(x)$ is strictly log-concave in x for all $0 < x \leqslant p\sqrt{3}$ provided $p > 0$. Finally, it follows from part (b) of Theorem 2.1 that the function $1/I_q(x)$ is strictly log-concave in x if $0 < x \leqslant 2b$ and $b - 1 \leqslant q \leqslant 0$. Since $2b \leqslant p\sqrt{3}$, the assertion follows. □

We close this paper with the following.

Conjecture. *The function $x \to \mathcal{I}_p(x)$ is strictly log-convex for all $p > -1$ and $x \in \mathbb{R}$ while the function $x \to I_p(x)$ is strictly log-concave for all $p > 0$ and $x > 0$.*

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